Problem 8.4.10

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02407 Stochastic Processes

We define a sequence of strictly increasing time points $\{t_i\}_{i\in\mathbb{N}_0}$ with $t_0 = 0$. Furthermore, we let $\{Z_t\}_{t\geq 0}$ be a geometric Brownian motion with drift parameter r and variance parameter σ^2 and define a discrete process $\{X_n\}_{n\in\mathbb{N}_0}$ given by $X_n = Z_{t_n} e^{-rt_n}$.

In this course, we define Martingales through the conditions stated on p. 72 in sec. 2.5.1. The first condition states that the absolute moment must always be finite. In order to show this, first note that geometric Brownian motions and exponential functions are strictly positive, which means that for the process $\{X_n\}$, the absolute moment coincides with the pure moment. Hence,

$$\mathbb{E}[|X_n|] = \mathbb{E}[|Z_{t_n} e^{-rt_n}|] = \mathbb{E}[Z_{t_n} e^{-rt_n}] = e^{-rt_n}\mathbb{E}[Z_{t_n}].$$

The mean value of a geometric Brownian motion might tend to infinity, but it is always finite, and we can conclude that the above expression is finite. Thus, the first condition from sec. 2.5.1 is satisfied. We now check the second condition from the mentioned section. We therefore consider the conditional expectation $\mathbb{E}[X_{n+1}|X_0, ..., X_n]$. Since X_n is a functional of Z_{t_n}, information about the state of X_n is actually information about the state of Z_{t_n}. As $\{Z_t\}$ is a geometric Brownian motion and consequently a Markov process, we can invoke the Markov property, which leads to

$$\mathbb{E}[X_{n+1}|X_0, ..., X_n] = \mathbb{E}[Z_{t_{n+1}} e^{-rt_{n+1}}|Z_{t_0}, ..., Z_{t_n}] = \mathbb{E}[Z_{t_{n+1}} e^{-rt_{n+1}}|Z_{t_n}] = e^{-rt_{n+1}}\mathbb{E}[Z_{t_{n+1}}|Z_{t_n}].$$

We can use eq. (8.50) to evaluate the conditional expectation in the last expression

$$\mathbb{E}[Z_{t_{n+1}}|Z_{t_n}] = Z_{t_n} e^{r(t_{n+1}-t_n)}.$$

In summary,

$$\mathbb{E}[X_{n+1}|X_0, ..., X_n] = e^{-rt_{n+1}}\mathbb{E}[Z_{t_{n+1}}|Z_{t_n}] = e^{-rt_{n+1}} Z_{t_n} e^{r(t_{n+1}-t_n)} = Z_{t_n} e^{-rt_n} = X_n,$$

which means that both conditions from sec. 2.5.1 are satisfied, and we conclude that $\{X_n\}$ is a Martingale.